Math 429 - Graded assignment (due May 7 at 08:15, in lecture)

1. Consider the 3-dimensional vector space $V = \mathbb{R}^3$.

(a) Show that the cross-product of vectors defines a Lie algebra structure on V. *(10 points)*

(b) Construct a Lie algebra isomorphism between V and one of the classical Lie algebras $\mathfrak{sl}_{n,\mathbb{R}}$, $\mathfrak{o}_{n,\mathbb{R}}$, $\mathfrak{sp}_{2n,\mathbb{R}}$ (for some n). Prove that what you construct is actually an isomorphism. *(10 points)*

2. (a) Prove that $\mathfrak{sl}_{2,\mathbb{F}_2}$ (traceless 2×2 matrices with coefficients in $\mathbb{Z}/2\mathbb{Z}$) is a nilpotent Lie algebra. *(10 points)*

(b) Prove that $\mathfrak{sl}_{3,\mathbb{F}_3}$ (traceless 3×3 matrices with coefficients in $\mathbb{Z}/3\mathbb{Z}$) is not a simple Lie algebra. *(10 points)*

3. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$, and consider the abstract Jordan decomposition

$$x = x_{ss} + x_n$$

for any $x \in \mathfrak{g}$, where $x_{ss}, x_n \in \mathfrak{g}$ are its semisimple and nilpotent parts, respectively.

(a) Prove that if $x, y \in \mathfrak{g}$ satisfy $[x, y] = 0$, then

$$(x + y)_{ss} = x_{ss} + y_{ss} \qquad \text{and} \qquad (x + y)_n = x_n + y_n \tag{1}$$

(10 points)

(b) Find a counterexample to (1) if $[x, y] \neq 0$ (you may choose $\mathfrak{g}$ and x, y as you wish). *(10 points)*

4. Both $\mathfrak{sl}_{2,\mathbb{C}}$ and $\mathfrak{gl}_{2,\mathbb{C}}$ act on the standard representation $\mathbb{C}^2$, as well as on its symmetric powers $S^n\mathbb{C}^2$. Let $\phi_n(x) : S^n\mathbb{C}^2 \to S^n\mathbb{C}^2$ denote the action of any $x \in \mathfrak{sl}_{2,\mathbb{C}}$ or $\mathfrak{gl}_{2,\mathbb{C}}$ in this representation.

(a) Explicitly calculate the symmetric invariant bilinear forms on both $\mathfrak{sl}_{2,\mathbb{C}}$ and $\mathfrak{gl}_{2,\mathbb{C}}$ given by

$$(x, y)_n = \mathrm{tr}_{S^n\mathbb{C}^2}(\phi_n(x)\phi_n(y))$$

(10 points)

(b) In the case of $\mathfrak{sl}_{2,\mathbb{C}}$, explain why $(\cdot, \cdot)_n$ must all be multiples of each other as n varies over $\mathbb{N}$. *(5 points)*

(c) In the case of $\mathfrak{gl}_{2,\mathbb{C}}$, show that $(\cdot,\cdot)_n$ is non-degenerate for all $n \in \mathbb{N}$; what does this say about $\mathfrak{gl}_{2,\mathbb{C}}$? *(5 points)*

5. Prove that the Killing form of a nilpotent Lie algebra (finite-dimensional over the field $\mathbb{C}$) is identically 0. *(10 points)*

6. Fix $1 \leq i < j \leq n$. By general theory, the Lie algebra homomorphism

$$\mathfrak{sl}_2 \to \mathfrak{sl}_n, \qquad E \rightsquigarrow E_\alpha,\ F \rightsquigarrow F_\alpha,\ H \rightsquigarrow H_\alpha$$

corresponding to the root $\alpha = e_i - e_j$ of $\mathfrak{sl}_n$ can be lifted to a Lie group homomorphism

$$SL_2(\mathbb{C}) \xrightarrow{\pi} SL_n(\mathbb{C}) \tag{2}$$

Find and prove an explicit formula for the homomorphism (2) (i.e. write down $\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ for any a, b, c, d such that $ad - bc = 1$). *(10 points)*